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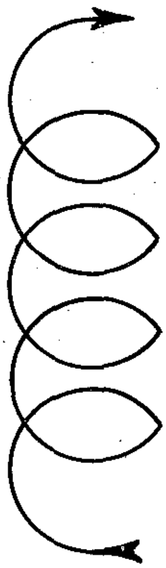

MAGNETOHYDRODYNAMIC WAVES

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1. INTRODUCTION

The aim of this paper is to study some special magnetohydrodynamic waves and their connection with the methods of their production, that is, the boundary conditions. Possible wave motions of a fluid form the underlying structure of the mathematical description; hence a knowledge of their behavior leads to a deeper understanding of fluid dynamical problems. Furthermore there is some evidence that these waves can be produced in the laboratory and may occur in nature. These waves occur in a model fluid which is an ordinary gas dynamic fluid endowed with a scalar electrical conductivity σ. In practice there is a fairly direct application to slightly ionized gases and to conducting liquids. However the general method of approach also applies to fully ionized plasmas described by continuum equations.

Many important results can be derived for an ideal conducting fluid $(\sigma \to \infty)$ which is analogous to the familiar ideal inviscid, non-heat conducting fluid $(\nu = k = 0)$. In particular various wave motions are studied in the presence of an initial magnetic field. In view of the important theorem that the total flux through a surface attached to the particles of an infinitely conducting fluid is constant, consideration has to be given to a method of getting flux lines into the fluid. From our

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point of view the ideal conducting fluid is an approximation to a fluid of high but finite conductivity. If sufficient time elapses before the start of the wave motion the field can be made to penetrate the fluid. Alternatively, the field can be introduced when the fluid is in a non-conducting state and then the fluid can be made conducting, for example by somehow raising the temperature.

In the following sections the production of various shock waves in a gas of finite constant conductivity will be studied. As a first step Section 2 discusses the production of shock waves by the motion of a piston into an ideal conducting gas. Such shock relations have been discussed in some detail in various references, for example in the relativistic case (Ref. 1) and the non-relativistic case (Ref. 2) but by methods somewhat different from those used here. In Section 3 a linearized problem is solved for the motion of a conducting piston into a gas of finite conductivity for the special case of an initial field transverse to the motion.

In the ideal conductor $(\sigma \rightarrow \infty)$ the shock wave is a discontinuity which carries a current sheet of infinite density. The solution of Section 3 shows how this discontinuity becomes asymptotically a diffused transition region. Section 4 returns to the ideal conducting gas and studies the properties of a special typical magnetohydrodynamic shock, the switch-on wave. This wave front is normal to field lines ahead but nevertheless turns the flow and the field. In Section 5 the production of a weak shock wave of switch-on type in a gas with finite electric conductivity is studied by means of the appropriate approximate equations. Such a wave can be produced by the sudden production of a current sheet.

In Section 6 some general comments are made in the light of the previous special solutions.

The magnetohydrodynamic approximation used in this paper is the conventional one in that displacement currents are neglected. Formally, the dielectric constant $\epsilon = 0$. This means that all wave speeds should be much less than the speed of light, and that the magnetic energy is much greater than the electric energy associated with the field. $\vec{E}$ may be eliminated in favor of $\vec{B}$. MKSQ units are used throughout.

2. SHOCK WAVES IN AN IDEAL CONDUCTING GAS

Consider gas initially at rest in $x > 0$ with conditions $(P_I, \rho_I, B_{I_x}, B_{I_y})$ and consider an infinite piston originally occupying $x < 0$ to be set in motion with a constant velocity u_F at time $t = 0$. If the gas has infinite conductivity and the solid piston has either zero or infinite conductivity there is no characteristic length. Hence the solution has conical similarity so that for the velocity in the x-direction $u(x, t) = f(\frac{x}{t})$ etc. One solution consists of a single discontinuity, a shock wave across which velocity and field jump, followed by a uniform state. In order to write down the shock relations between the final and initial states it is convenient to represent the effect of the currents which flow in the shock front and at the piston face by the relevant part of the Maxwell stress tensor

$$T = \frac{1}{\mu}\begin{pmatrix} \frac{B_x^2 - B_y^2 - B_z^2}{2} & B_xB_y & B_xB_z \\ B_xB_y & \frac{B_y^2 - B_x^2 - B_z^2}{2} & B_yB_z \\ B_xB_z & B_yB_z & \frac{B_z^2 - B_x^2 - B_y^2}{2} \end{pmatrix} \quad (2\text{-}1)$$

Also, for purposes of the energy equation, notice that the electromagnetic energy per volume associated with the field is

$$W = \frac{B^2}{2\mu} = \frac{B_x^2 + B_y^2 + B_z^2}{2\mu} \quad (2\text{-}2)$$

The shock relations are then (2-3):

continuity $\qquad \frac{\rho_I}{\rho_F} = 1 - \frac{u_F}{c}$

x-momentum $\qquad P_F + \frac{B_{F_y}^2 - B_{F_z}^2 - B_{F_x}^2}{2\mu} - P_I - \frac{B_{I_y}^2 - B_{I_x}^2}{2\mu} = \rho_I u_F c$

y-momentum $\qquad -\frac{1}{\mu} B_{F_y} B_{F_x} + \frac{1}{\mu} B_{I_y} B_{I_x} = \rho_I v_F c$

z-momentum $\qquad -\frac{1}{\mu} B_{F_z} B_{F_x} = \rho_I w_F c$

energy

$$\rho_I c \left\{\frac{1}{\gamma-1}\left(\frac{P_F}{\rho_F} - \frac{P_I}{\rho_I}\right) + \frac{u_F^2 + v_F^2 + w_F^2}{2}\right\}$$

$$+ (c - u_F)\left\{\frac{B_{F_x}^2 + B_{F_y}^2 + B_{F_z}^2}{2\mu}\right\} - c\,\frac{B_{I_x}^2 + B_{I_y}^2}{2\mu}$$

$$= \left(P_F + \frac{B_{F_y}^2 + B_{F_z}^2 - B_{F_x}^2}{2\mu}\right)u_F - \frac{1}{\mu} B_{F_y} B_{F_x} v_F - \frac{1}{\mu} B_{F_z} B_{F_y} w_F$$

induction $$v_F B_{F_x} + (c - u_F) B_{F_y} = c B_{I_y}$$

induction $$w_F B_{F_x} + (c - u_F) B_{F_z} = 0$$

continuity for B $$B_{I_x} = B_{F_x}$$

The continuity equation is derived by noting that the mass between the shock and piston at time t occupies a volume $(c - u_F)t$ with a density ρ_F but at $t = 0$ occupied a volume ct with a density ρ_I. The momentum equations balance the impulse applied to the gas by the normal and tangential stresses at the piston with the change in momentum produced by the motion of the shock. The energy equation considers all the gas and that part of the field between the piston and the shock as a thermodynamic system. The left-hand side represents the change in internal and kinetic energy of the perfect gas as well as the change in field energy while the right-hand side represents the rate of working of the piston on the gas. There is no Poynting flux for this system since, to prevent infinite currents from flowing, the electric field is zero in any system of coordinates fixed in the fluid. The induction equations are derived by noting that there is no flux through rectangular contours parallel to the $(x-y)$ and $(x-z)$ planes at $t = 0$. If these contours are attached to the fluid particles they rotate as the shock wave progresses.

From this point of view we are given $(P_I, \rho_I, \vec{B}_I, u_F)$ and must find $(P_F, \rho_F, \vec{B}_F, v_F, w_F, c)$. The main unknown is the shock speed c; after

c is determined as a function of the given quantities it is easy to determine all the other unknowns. Before determining c consider the consequences of the tangential momentum and the induction equations:

$$B_{F_y} = B_{I_y} \frac{c^2-b_x^2}{c^2-cu_F-b_x^2}, \quad B_{F_z} = 0 \tag{2-4}$$

where $b_x^2 = [\text{Alfvén speed}]^2 = \dfrac{B_{I_x}^2}{\rho_I \mu}$

$$v_F = -u_F \frac{b_y b_x}{c^2-cu_F-b_x^2}, \quad w_F = 0 \tag{2-5}$$

These results are valid if

$$c^2-cu_F-b_x^2 \neq 0 \tag{2-6}$$

(2-4) shows how the transverse field is increased in proportion to its original value and (2-5) shows how the flow is turned at the same time.

The general equation for the wave speed c can be obtained from the energy equation by using the induction, momentum and continuity equations to eliminate P_F, ρ_F, and B_{F_y}

$$\begin{aligned}(c^2-\tfrac{\gamma+1}{2}cu_F-a_I^2)(c^2-cu_F-b_x^2)^2 \\ = c\,b_y^2\left\{c^3-\tfrac{\gamma+2}{2}u_Fc^2-(b_x^2-\tfrac{\gamma}{2}u_F^2)c+\tfrac{\gamma+1}{2}u_Fb_x^2\right\}\end{aligned} \tag{2-7}$$

This sixth degree equation expresses the possible roots $(\frac{c}{a_I})$ as a function of the three parameters $\frac{u_F}{a_I}$, $\frac{b_x}{a_I}$, $\frac{b_y}{a_I}$. A general discussion is not possible here but we can discuss several special cases. If $b_y = 0$ the shock is normal to the field lines and

$$c^2 - \frac{\gamma+1}{2} c u_F - a_I^2 = 0 \quad \text{or} \quad c = \frac{\gamma+1}{4} u_F + \sqrt{\frac{(\gamma+1)^2}{16} u_F^2 + a_I^2} \qquad (2\text{-}8)$$

This is an ordinary gasdynamic shock which for a weak wave $(u_F \to 0)$ travels with the sound speed a_I; for a strong shock $c \to \frac{\gamma+1}{2} u_F$. In Section 4 another solution with a shock normal to the field lines will be discussed. A complete discussion can be given for weak waves where $u_F \to 0$. Then (2-7) becomes

$$(c^2 - b_x^2)\left\{c^4 - c^2(a_I^2 + b_x^2 + b_y^2) + a_I^2 b_x^2\right\} = 0 \qquad (2\text{-}9)$$

The wave speeds for weak waves given by (2-9) are identical with those computed on the basis of the theory of characteristics (Ref. 2). The root $c = \pm b_x$ represents transverse waves, analogous to Alfvén waves in an incompressible flow. The vanishing of the other bracket corresponds to the fast and slow coupled waves. These are the magneto-hydrodynamic generalizations of sound waves and they show the strong anisotropy typical of magneto-hydrodynamic waves (see Fig. 1). The faster wave travels along the field lines with the speed a_I or b_x whichever is greater and across the field lines with the combined speed:

$$c = \sqrt{a_I^2 + b_y^2} = c_o \qquad (2\text{-}10)$$

This result is a special example of a more general result which shows how gas and magnetic pressure combine. A weak wave of the type described by (2-10) is studied in the next section to see the effect of finite conductivity and the boundary conditions.

3. LINEARIZED WAVE FOR FINITE σ; PISTON PROBLEM

For the piston problem consider the gas initially at rest in the presence of a field $B^*_{I_y}$ parallel to the piston face and consider the piston suddenly set into motion with the speed u_F (see Fig. 2). When gas with a finite σ is considered a characteristic length is introduced. Hence to write down the equations of motion consider

$$T^* = \frac{1}{\mu\sigma a_I^2} = \text{characteristic time} \tag{3-1a}$$

$$L^* = \frac{1}{\mu\sigma a_I} = \text{characteristic length} \tag{3-1b}$$

and the following dimensionless variables

$$x = \frac{x^*}{L^*}, \quad t = \frac{t^*}{T^*}, \quad \vec{q} = \frac{\vec{q}^{\,*}}{a_I}, \quad \vec{B} = \frac{\vec{B}^*}{|\vec{B}_I^*|}, \quad \vec{E} = \frac{\vec{E}^*}{|\vec{B}_I^*|\, a_I}$$

$$P = \frac{P^*}{P_I^*}, \quad \rho = \frac{\rho^*}{\rho_I^*}, \quad S = \frac{S^* - S_I^*}{c_v}, \quad \vec{j} = \frac{\vec{j}^{\,*}}{\sigma a_I |\vec{B}_I^*|}$$

The starred variables are quantities with physical dimensions. The dimensionless equations of motion and Maxwell equations may be written

$$\rho_t + \operatorname{div} \rho\vec{q} = 0 \tag{3-2a}$$

$$\rho(\vec{q}_t + \vec{q}\cdot\nabla\vec{q}) = -\frac{1}{\gamma}\nabla P + \beta^2(\vec{j}\times\vec{B}) \tag{3-2b}$$

$$P(S_t + \vec{q}\cdot\nabla S) = \gamma(\gamma - 1)\beta^2 j^2 \tag{3-2c}$$

$$P = \rho^\gamma e^S \tag{3-2d}$$

$$\vec{B}_t = -\operatorname{curl}\vec{E} \tag{3-2e}$$

$$\vec{j} = \vec{E} + \vec{q} \times \vec{B} = \mathrm{curl}\,\vec{B} \qquad \text{in gas} \tag{3-2f}$$

$$\vec{j} = \frac{1}{\lambda}(\vec{E} + \vec{q} \times \vec{B}) = \mathrm{curl}\,\vec{B} \qquad \text{in solid} \tag{3-2g}$$

The equations contain the parameters

$$\lambda = \frac{\sigma}{\sigma_P} = \frac{\text{conductivity of gas}}{\text{conductivity of piston}} \tag{3-3a}$$

$$\beta = \frac{b}{a} = \frac{\text{Alfvén speed}}{\text{initial sound speed}}\,, \quad b = \frac{|\vec{B}_I^*|}{\rho_I^* \mu} \tag{3-3b}$$

and the motion of the piston introduces the parameter

$$M = \frac{u_F}{a_I} = \text{piston Mach number}$$

The boundary conditions at the piston face are

$$(\vec{q})_1 = q_1 = M \tag{3-4a}$$

$$(\vec{B})_{\text{solid}} = (\vec{B})_{\text{gas}} \tag{3-4b}$$

$$(\vec{E} + \vec{q} \times \vec{B})_{\text{tang. solid}} = (\vec{E} + \vec{q} \times \vec{B})_{\text{tang. gas}}^{*} \tag{3-4c}$$

The continuity of tangential components of $\vec{B}$ and $\vec{E}$ follow from the requirements of finite current density $\vec{j}$ and finite fields $\vec{B}$. For weak waves the various dimensionless quantities may be expanded in terms of the small parameter M as follows

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*for the moving surface this follows from the integral form

$$\frac{d}{dt}\iint \vec{B} \cdot d\vec{A} = -\oint (\vec{E} + \vec{q} \times \vec{B}) \cdot d\vec{s}$$

$$\vec{q}(x, t; M) = Mu(x, t)\vec{i}_1 + \ldots \tag{3-5a}$$

$$P = 1 + Mp(x, t) + \ldots \tag{3-5b}$$

$$\rho = 1 + M\rho^+(x, t) + \ldots \tag{3-5c}$$

$$\vec{B} = (1 + Mb_2(x, t)\vec{i}_2 + \ldots \tag{3-5d}$$

In this problem we have only to deal with a velocity in the x-direction, $\vec{B}$ in the y-direction, and currents in the z-direction.

The linearized equations of motion are

$$\left.\begin{array}{ll} \rho_t^+ + u_x = 0 & (3\text{-}6a) \\ u_t = -\frac{1}{\gamma}\, p_x - \beta^2 b_{2_x} & (3\text{-}6b) \\ b_{2_t} = b_{2_{xx}} - u_x & (3\text{-}6c) \\ p = \gamma\rho^+ & (3\text{-}6d) \end{array}\right\} \quad \text{in gas } x > 0$$

$$b_{2_t} = \lambda b_{2_{xx}} \qquad (3\text{-}7) \qquad \text{in solid } x < 0$$

The boundary conditions are applied at $x = 0$ consistent with the approximation. By elimination from (3-6) we note

$$u_{tt} - u_{xx} = -\beta^2 b_{2_{xt}} \tag{3-8a}$$

$$b_{2_{xx}} - b_{2_t} = u_x \tag{3-8b}$$

(3-8a) shows how sound waves are generated by the presence of a field and (3-8b) shows how the flow generates a diffusing field. The ordinary sound waves with speed a_I are the real characteristics of the system (3-6) but as will be seen disturbances associated with these waves damp out with time; the main disturbance will eventually be associated with the sub-characteristics, speed $c_o = \sqrt{a_I^2 + b^2}$.

The linearized boundary conditions are now

$$u(0, t) = 1 \tag{3-9a}$$

$$b_2(0+, t) = b_2(0-, t) \tag{3-9b}$$

$$b_{2_x}(0+, t) = \lambda b_{2_x}(0-, t) \tag{3-9c}$$

A representation of the solution can be obtained by application of Laplace Transform. Solutions of (3-7) which die out at $x \to -\infty$ yield one arbitrary constant while solutions of the system (3-8) which die out as $x \to +\infty$ yield two arbitrary constants. The three arbitrary constants are determined from the three boundary conditions (3-9). The Laplace Transform of the solution is $(x > 0)$

$$\overline{u}(x;s) = \frac{1}{sQ}\left\{(\alpha_2^2 - s^2)\left(\frac{1}{\alpha_2}\sqrt{\frac{s}{\lambda}} - 1\right)e^{\alpha_1 x} - (\alpha_1^2 - s^2)\left(\frac{1}{\alpha_1}\sqrt{\frac{s}{\lambda}} - 1\right)e^{\alpha_2 x}\right\} \tag{3-10a}$$

$$\overline{b_2}(x;s) = -\frac{s}{Q}\left\{\frac{1}{\alpha_1}\left(\frac{1}{\alpha_2}\sqrt{\frac{s}{\lambda}} - 1\right)e^{\alpha_1 x} - \frac{1}{\alpha_2}\left(\frac{1}{\alpha_1}\sqrt{\frac{s}{\lambda}} - 1\right)e^{\alpha_2 x}\right\} \tag{3-10b}$$

where

$$Q = (\alpha_2^2 - s^2)\left(\frac{1}{\alpha_2}\sqrt{\frac{s}{\lambda}} - 1\right) - (\alpha_1^2 - s^2)\left(\frac{1}{\alpha_1}\sqrt{\frac{s}{\lambda}} - 1\right)$$

$$\alpha_{1,2}^2 = \frac{1}{2}[s^2 + s(1+\beta^2)] \pm \sqrt{\frac{1}{4}[s^2 + s(1+\beta^2)]^2 - s^3}$$

$$\text{Rl } \alpha_{1,2} > 0 \text{ on contour of integration}$$

The behavior for large time is, in a certain sense, determined by the behavior of the transforms near $s = 0$ and we note

$$a_1 \to -\sqrt{s}\sqrt{1+\beta^2} \tag{3-11a}$$

$$a_2 \to -\frac{s}{\sqrt{1+\beta^2}} + \frac{1}{2}\beta^2 \frac{s^2}{(1+\beta^2)^{5/2}} + \ldots \tag{3-11b}$$

The asymptotic behavior at the piston face $(x = 0)$ is easily determined from the transforms (3-10) and from (3-11), by letting $s \to 0$:

$$b_2(0+, t) \cong \frac{1}{\sqrt{1+\beta^2}} \left\{ \frac{1}{1 + \frac{1}{\sqrt{\lambda(1+\beta^2)}}} \right\} \tag{3-12}$$

$$p(0+, t) \cong \frac{\gamma}{\sqrt{1+\beta^2}} \left\{ \frac{1 + \sqrt{\frac{1+\beta^2}{\lambda}}}{1 + \frac{1}{\sqrt{\lambda(1+\beta^2)}}} \right\} \tag{3-13}$$

(3-12) and (3-13) are proportional to the perturbation Maxwell stress and perturbation gas pressure at the piston; they show how these quantities change depending on the ratio λ of electric conductivity in the gas to that in the piston. The Maxwell stress is of course a measure of the current flowing in the piston; alternatively the magnetic force on the piston could be computed from the product of this current and the original field. The total perturbation stress, however, is independent of λ, because as will be seen shortly, the motion of the piston produces a wave motion in the gas independent of λ. Addition of (3-12) and (3-13) shows

$$\text{perturbation stress} = M(P_I^* p + \frac{B_{I_y}^{*2}}{\mu} b_2) \cong \rho_I^* u_F \sqrt{a_I^2 + b_y^2} \qquad (3\text{-}14)$$

(3-14) is the same formula as for the pressure increment associated with the passage of a sound wave if the acoustic speed is replaced by the combined speed c_o. Thus, even in the gas of finite conductivity σ the solution for infinite conductivity discussed in the previous section has significance for large times, that is, for $t^* \gg T^* = \frac{1}{\mu \sigma a_I^2}$.

The form of the waves can easily be obtained by a crude asymptotic method. For large t replace $\alpha_{1,2}$ by the approximation (3-11) and notice that $\alpha_1 \sim \sqrt{s}$ corresponds to a diffusion about the origin. Thus the wave is associated with α_2, the first term of which corresponds to a discontinuity at $x^* = c_o t^*$. Thus, study part of (3-10a),

$$u_t^{(2)}(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st+\alpha_2(s)x} \phi(s) ds \qquad (3\text{-}15)$$

by using (3-11) and then applying the method of steepest descent. The results obtained this way are valid for large t:

$$u(x, t) \cong \frac{1}{2} \operatorname{erfc} \frac{(x - \sqrt{\beta^2+1}\, t)(\beta^2 + 1)^{3/4}}{\beta \sqrt{2x}} \qquad (3\text{-}16)$$

$$b_2(x, t) \cong \frac{1}{2\sqrt{1+\beta^2}} \operatorname{erfc} \frac{(x - \sqrt{\beta^2+1}\, t)(\beta^2 + 1)^{3/4}}{\beta \sqrt{2x}}$$

$$- \frac{1}{\sqrt{1+\beta^2}} \frac{\sqrt{\lambda(1+\beta^2)}}{1 + \frac{1}{\sqrt{\lambda(1+\beta^2)}}} \operatorname{erfc} \frac{(1+\beta^2)x}{2\sqrt{t}} \qquad x > 0 \qquad (3\text{-}17a)$$

$$\cong \frac{1}{\sqrt{1+\beta^2}} \frac{1}{1 + \frac{1}{\sqrt{\lambda(1+\beta^2)}}} \operatorname{erfc} \frac{-x}{2\sqrt{\lambda t}} \qquad x < 0 \qquad (3\text{-}17b)$$

The solutions show a wave front diffusing about $x = \sqrt{\beta^2+1}\ t$ which is $x^* = c_o t^*$ the discontinuity of the ideal conducting gas. Any disturbance associated with the ordinary sound wave is damped out exponentially. Remembering that the current density

$$\vec{j} = M b_{2_x} \vec{i}_3 \tag{3-18}$$

we see from (3-17) peaks of current flowing in the rapid transition zone at the wave front and in the neighborhood of the piston face. A sketch of the form of the solution is given in Fig. 3. An integration of (3-18) from $-\infty$ to ∞ shows that the total current which flows must be zero. The current at the shock front is equal and opposite to the current which flows near the piston face.

For small values of t the character of the solution is quite different; diffusion about the origin and the ordinary sound wave are the important features.

4. SWITCH-ON SHOCK WAVES IN IDEAL CONDUCTING GAS

In the derivation of the solution (2-7) for the shock waves in the ideal conductor the restriction (2-6) was imposed that $c^2 - cu_F - b_x^2 \neq 0$. It is natural to enquire if solutions satisfying the shock wave relations (2-3) can be found if

$$c(c - u_F) = b_x^2 \tag{4-1}$$

As implied by (2-4) this relation can only be satisfied when the initial field parallel to the shock is zero

$$B_{I_y} = B_{I_z} = 0 \tag{4-2}$$

and in this case (4-1) is a necessary condition that the induction and tangential momentum equations can be satisfied. In fact these equations reduce to

$$\frac{v_F}{w_F} = \frac{B_{F_y}}{B_{F_z}} \tag{4-3}$$

$$\frac{B_{I_x}^2}{\mu^2}\left(B_{F_y}^2 + B_{F_z}^2\right) = \rho_I c^2 (v_F^2 + w_F^2) = \rho_I c^2 q_F^2 \quad \text{(say)} \tag{4-4}$$

That is, if such a wave exists the flow direction and field direction are the same in a transverse plane as indicated by (4-3) and the magnitude of the transverse velocity is proportional to the field magnitude by (4-4). Such a shock wave which produces a transverse field behind the shock when there is no transverse field ahead is called a "switch-on" shock wave (cf. Ref. 2). As far as the shock relations (2-3) are concerned there is of course no preferred direction so that only q_F^2 will enter further discussions; a one parameter family of solutions will be obtained.

The existence of the switch-on shock depends on whether or not the energy equation can be satisfied; this equation now determines a relation between c and q_F. If the energy equation of (2-3) is studied using (4-1) to (4-4) and the momentum and continuity equations it is found that

$$q_F^2 = 2\frac{u_F}{c}\left\{c^2 - \frac{\gamma+1}{2}\, cu_F - a_I^2\right\} \tag{4-5a}$$

$$= 2(b_x^2 - a_I^2)\frac{u_F}{c} - (\gamma-1)u_F^2 \tag{4-5b}$$

The second relation shows us that it is necessary that $b_x > a_I$ if a switch-on wave is to exist while the first relation shows us that the speed of the switch-on wave must be greater than that of the corresponding ordinary gasdynamic shock (same u_F, cf. (2-8)). The switch-off wave has zero strength both when $u_F \to 0$ and when the speed of the switch-on wave is equal to that of the corresponding shock; for u_F larger than that critical value the solution no longer exists. Fig. 4 illustrates the region in which switch-on waves can occur by plotting (2-8) and (4-1) for $b_x = \sqrt{2}\, a_I$. A similar analysis can be applied to switch-on waves.

In order to study these waves in gas with finite σ a weak wave approximation will again be necessary.

In this case however the relations (4-5) show us that the velocity perturbation associated with the wave in the transverse and x-direction are of different orders of magnitude.

$$\begin{aligned} &u_F \to 0 \\ &c \to b_x \\ &q_F \to \sqrt{2u_F}\sqrt{\frac{b_x^2 - a_I^2}{b_x}} \end{aligned} \tag{4-6}$$

The fact that $q_F = O(\sqrt{u_F})$ must be taken into account if a proper description is to be given of the waves in a fluid with finite σ. The procedure is sketched in the next section.

5. WEAK SWITCH-ON WAVES IN A GAS WITH FINITE σ.

In accordance with the remarks of the previous section the following form of expansion is assumed for the weak waves (flow in x, y plane)

$$\vec{q} = J^2 u(x, t)\,\vec{i}_1 + Jv(x, t)\,\vec{i}_2$$

$$P = 1 + J^2 p$$

$$\rho = 1 + J^2 \rho^+ \qquad (5\text{-}1)$$

$$S = 1 + J^2 S^+$$

$$\vec{B} = (1 + J^2 b_1)\,\vec{i}_1 + Jb_2 \vec{i}_2$$

J is a small parameter to be defined. If the expansion (5-1) is substituted into the dimensionless equations of motion (3-2) first approximation equations are found:

$$\rho_t^+ + u_x = 0 \qquad (5\text{-}2a)$$

$$u_t = -\frac{\partial}{\partial x}\left(\frac{p}{\gamma} + \beta\frac{b_2^2}{2}\right) \qquad (5\text{-}2b)$$

$$v_t = \beta^2 b_{2_x} \qquad (5\text{-}2c)$$

$$p - \gamma p^+ = S^+ \qquad (5\text{-}2d)$$

$$S_t^+ = \gamma(\gamma - 1)\beta^2 b_{2_x}^2 \qquad (5\text{-}2e)$$

$$b_{2_t} = (b_{2_x} + v)_x \qquad (5\text{-}2f)$$

In this system notice that the transverse momentum and induction equations are uncoupled so that (v, b_2) may be determined separately.

The $b_2(x, t)$ so found then acts as a forcing function to produce a flow in the x-direction, and pressure and density changes. These weak waves are not isentropic but the Joule heating is important. The system (5-2) can also be expressed as a reduced system by elimination:

$$b_{2_{tt}} - \beta^2 b_{2_{xx}} = b_{2_{xxt}} \tag{5-3a}$$

$$v_t = \beta^2 b_{2_x} \tag{5-3b}$$

$$u_{tt} - u_{xx} = - \frac{\partial^2}{\partial x \partial t} \left\{ \frac{S^+}{\gamma} + \beta^2 \frac{b_2^2}{2} \right\} = - F_{xt} \text{ (say)} \tag{5-3c}$$

(5-3a) is identical with the equation for sound waves in a viscous gas which has been studied extensively (Ref. 3 and Ref. 4). The left-hand side would represent a wave travelling at the Alfvén speed $x^* = b_x t^*$ as in the ideal conductor while the right-hand side causes a diffusion of this wave front, just as in Section 3. (5-3b) associates a transverse acceleration of the fluid with the current which flows while (5-3c) shows how the Joule heating and Maxwell stress combine to produce waves travelling at the ordinary sound speed $x^* = a_1 t^*$.

Now the small parameter J must be linked up with the necessary boundary conditions to produce a weak switch-on wave. If one tries to solve a piston problem and to identify $J^2 \to M$ it is found that the boundary conditions (derived for example by integrating (3-2f) and the induction equation in the solid over a small area crossing the piston face $x = 0$) are homogeneous in (b, v). The problem has only the solution zero, which is gratifying as there was no distinguished direction anyway. This means that when a piston is pushed into conducting gas with the field lines normal to the piston only the ordinary gasdynamic shock or sound wave corresponding to (2-8) is produced.

On the other hand if a current sheet in the z-direction is turned on at the origin a b_2 field exists in its immediate neighborhood and a switch-on wave is produced if $b_x > a_I$. For such a current sheet the dimensionless current density at the origin is represented by

$$\vec{j} = J\delta(x)H(t) \tag{5-4}$$

and the total current flowing per length of sheet is $Ja_I|B^*_{I_x}|\sigma L^*$ or $J\frac{|B^*_{I_x}|}{\mu}$. The conducting gas is assumed to extend to infinity on both sides of the sheet so that the boundary conditions are

$$b_2(0+, t) - b_2(0-, t) = H(t) \tag{5-5a}$$

$$(b_{2_x} + v)_{0+} = (b_{2_x} + v)_{0-} \tag{5-5b}$$

$$u(0, t) = 0 \tag{5-5c}$$

The solution to (5-3a) and (5-3b) under the boundary conditions (5-5a, b) may be represented, using Laplace Transforms, by

$$2b_2(x, t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st - \frac{sx}{\sqrt{s+\beta^2}}}\frac{ds}{s} \qquad x > 0 \tag{5-6}$$

The asymptotic behavior of this integral may be derived by using the method of Section 3 or by taking the results over from Ref. 3 or 4 where the same integral was studied in some detail. As expected, for large times we find a wave front of b_2 diffusing about the wave traveling at Alfvén speed, and for small times a pure diffusion process.

$$2b_2(x,t) \,\tilde{=}\, \frac{1}{2}\,\mathrm{erfc}\,\frac{x-\beta t}{\sqrt{2t}}\,\left(1 + O\left(\frac{1}{\beta^2 t}\right)\right) \tag{5-7a}$$

$$\doteq \mathrm{erfc}\,\frac{x}{\sqrt{2t}}\left(1 + O(\sqrt{t})\right) \tag{5-7b}$$

For large t also (5-3b) shows

$$v(x,t) \,\tilde{=}\, -\beta b_2(x,t) \qquad x > 0 \tag{5-8}$$

Thus these are again current sheets near the wave fronts $x = \pm\beta t$ with a total current flowing equal and opposite to that in the current sheet. The general shape of this solution for large t is indicated in Fig. 5.

In order to obtain the associated pressure and density changes and the flow in the x-direction (5-2e) must first be integrated to obtain the entropy change and then (5-3c) must be solved with the known right-hand side and the boundary condition (5-5c). The solution to (5-3c) can be expressed as the integral over certain waves which can send signals to the (x, t) point in question:

$$u(x,t) = \frac{1}{2}\int_0^{t-x} F_x(\xi, t-x-\xi)d\xi - \frac{1}{2}\int_0^{x} F_x(\xi, t-x+\xi)d\xi$$

$$-\frac{1}{2}\int_x^{x+t} F_x(\xi, t-\xi+x)d\xi \qquad t > x \tag{5-9}$$

$$= \frac{1}{2}\int_{x-t}^{x} F_x(\xi, t-x+\xi)d\xi - \frac{1}{2}\int_x^{x+t} F_x(\xi, t-\xi+x)d\xi \qquad x > t$$

An $(x-t)$ diagram of the lines over which the integration is carried out is presented in Fig. 6. The function $F(x, t)$ is too complicated to make an exact integration feasible. A very crude integration may provide a qualitative picture of what is taking place. For purposes of evaluating the integrals in (5-9) the asymptotic behavior for large t is used and erfc is replaced by a step function, the derivative of erfc by a delta function where possible. This process yields

$$S^{+} \rightarrow \frac{(\gamma-1)\beta}{4} \frac{1}{\sqrt{\pi t}} \delta(x-\beta t) \tag{5-10}$$

That is, the entropy production by Joule heat is confined to the neighborhood of the switch-on wave front and becomes negligible for large times compared to the effect of Maxwell stress. Thus

$$F_x(x, t) \rightarrow -\beta^2 \frac{\delta(x-\beta t)}{8} \tag{5-11}$$

The integration of (5-9) now shows a constant u following the switch-on wave and an increased pressure between $x = t$ and $x = \beta t$. The gas is brought to rest by an expansion sound wave at $x = t$ and the entire region $0 < x < t$ is at rest. A sketch of the flow is given in Fig. 5. The value of u computed by this crude method agrees with that predicted for a weak switch-on wave in an ideal conducting gas. However, the effect of finite conductivity should produce some flow behind the expansion wave.

6. COMMENTS

A fairly clear picture has been given of the effect of a current sheet which produces a switch-on shock followed by an expansion fan if $b_x > a_I$. In order to understand what happens if a current sheet is discharged when $b_x < a_I$ consider the following limiting case. Let $b_x = 0$ $(\vec{B}_I^* \equiv 0)$ and consider the ideal conducting gas $\sigma \rightarrow \infty$ with a current sheet being discharged at the origin as before. Now since there was no flux originally in the gas none can be introduced. However some field is produced by the current sheet which means that the gas must be pushed away from the sheet by the magnetic pressure in the neighborhood of the sheet. This magnetic pressure acts exactly like a piston and an ordinary shock is sent out into the gas. The condition which determines the speed of the interface is exactly that the gas pressure produced by the shock should balance the magnetic pressure. This solution only exists, of course, if $P_F > P_I$. Using the gasdynamic shock relations

$$P_F = P_I + \rho_I u_F c = \frac{B_y^2}{2\mu} = \frac{I^2}{8\mu} \tag{6-1}$$

where

I = current per length in sheet

and

$$c^2 - \frac{\gamma+1}{2} c u_F - a_I^2 = 0 \tag{6-2}$$

These formulas enable u_F to be expressed as a function of the current flowing in the current sheet. Note that the magnetic pressure acts directly as a piston pressure; this would produce a much stronger shock than if the equivalent pressure were used in a shock tube. Note also that in this flow current sheets totaling a current equal and opposite to that in the current sheet at the origin flow in the gas-field interfaces. For strong waves (6-1) and (6-2) simplify to

$$\frac{2c}{\gamma+1} = u_F = \frac{1}{\sqrt{\gamma+1}}\, b_y, \quad b_y = \frac{B_y}{\sqrt{\rho_I \mu}} \tag{6-3}$$

Now when a small B_{I_x} is introduced into the problem it is to be expected that the same gasdynamic shock would be produced but would be followed by a magnetic switch-on expansion fan which would accomplish some of the turning of the field and flow. As b_x is increased the same pattern persists until $b_x > a_I$ when the switch-on wave can run ahead of the gasdynamic shock and the flow discussed in Sections 5 and 6 is produced. The description here is not complete but serves only to illustrate some of the features of waves produced by current sheets.

In order to realize waves similar to these one-dimensional waves in the laboratory a means of letting the necessary currents flow must be provided. This can be achieved by utilizing axial symmetry. A plane wave can be sent down a narrow annular tube in which case the currents run in circles around the annulus. Or these plane wave solutions can be considered as approximations to cylindrical waves at some distance from the axis; currents again flow in circles around the axis.

Non-linear effects can be considered in a qualitative way for magneto-hydrodynamic waves as for ordinary gasdynamic waves. Finite conductivity σ tends to diffuse the sharp wave fronts but whenever there is any u component following the wave non-linear transport (uu_x, uB_x, etc.) tends to steepen the wave front. The net effect is to allow the wave to approach a steady state forming a wave whose thickness is less than that predicted by linearized theory. In fact, for the waves considered here in Section 3 the thickness is $O\left(\frac{1}{\sqrt{\sigma}}\right)$ but for non-linear waves the thickness is $O(\frac{1}{\sigma})$. This last estimate follows directly from the thermodynamic reasoning balancing the entropy increase in the wave with the dissipation due to Joule heating. Some steady state shock wave solutions of this type have been computed by Marshall (Ref. 5). It is important of course in carrying out such calculations to incorporate the variations of σ with temperature. Marshall's solutions sometimes showed a rapid transition zone, like an ordinary shock wave, in the interior of the relatively wide magnetic shock zone. This is another effect of non-linearity which allows the damped sound wave of Section 3 to catch up and appear in the steady-state within the wave front diffusing about the combined speed. This connection has been pointed out by Whitham in a recent paper (Ref. 6) where he gives an elegant discussion of the general types of linearized equations which occur for finite σ, and their asymptotic behavior as well as some discussion of non-linear effects.

7. REFERENCES

1. de Hoffman, F. and E. Teller, Magneto-Hydrodynamic Shocks. Phys. Rev. (2), 80, 692-703 (1950).

2. Friedrichs, K. O., Magnetohydrodynamic Waves. Los Alamos Report (1954).

3. Lagerstrom, P. A., J. D. Cole and L. Trilling, Problems in the Theory of Viscous Compressible Fluids. Monograph, California Institute of Technology (1949).

4. Wu, T. Y., Small Perturbations in the Unsteady Flow of a Compressible, Viscous and Heat-Conducting Fluid. J. Math and Physics, 35, 1, 13-27 (1956).

5. Marshall, W., The Structure of Magneto-Hydrodynamic Shock Waves. Proc. Roy. Soc. London, 233, 367-376 (1955).

6. Whitham, G. B., Some Comments on Wave Propagation With Application to Hydrodynamics. Private communication (1958) to be published in Comm. Pure Applied Math.

$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \qquad b = \sqrt{\frac{B^2}{\rho\mu}}$

$b = \sqrt{2}a$

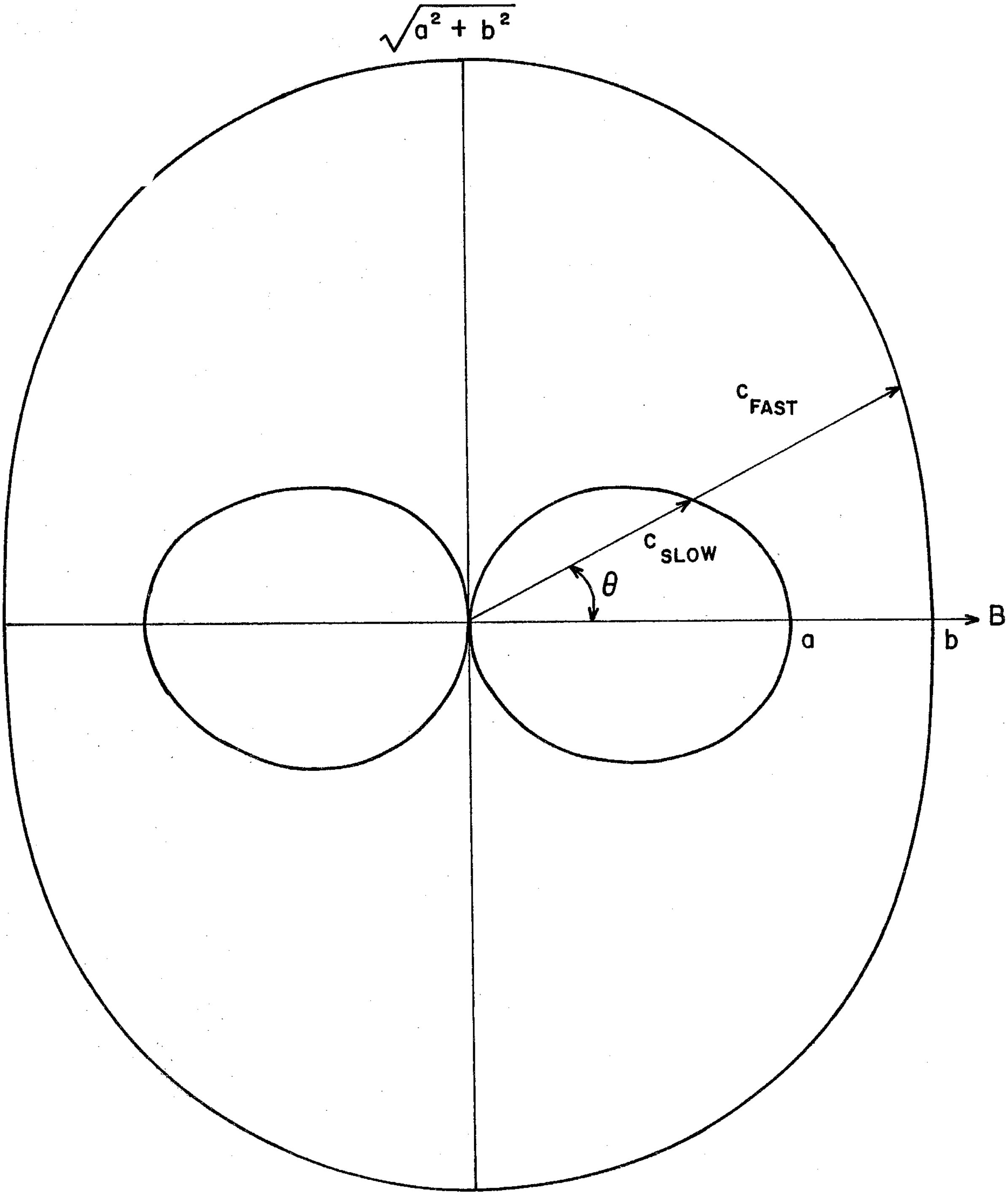

FIG. 1

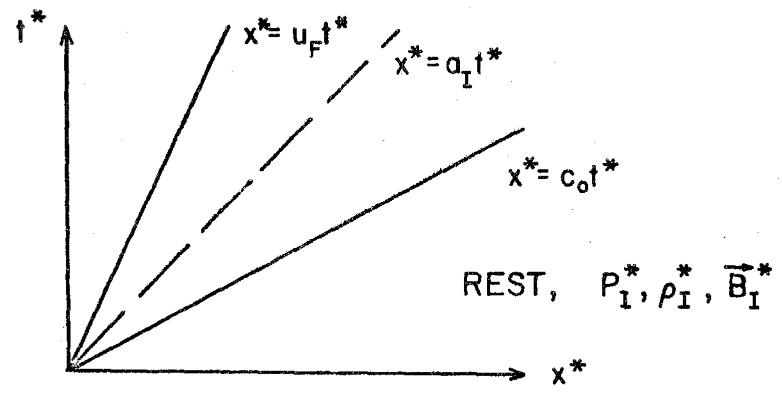
t^*
$x^* = u_F t^*$
$x^* = a_I t^*$
$x^* = c_0 t^*$
REST, $P_I^*, \rho_I^*, \vec{B}_I^*$
x^*

FIG. 2

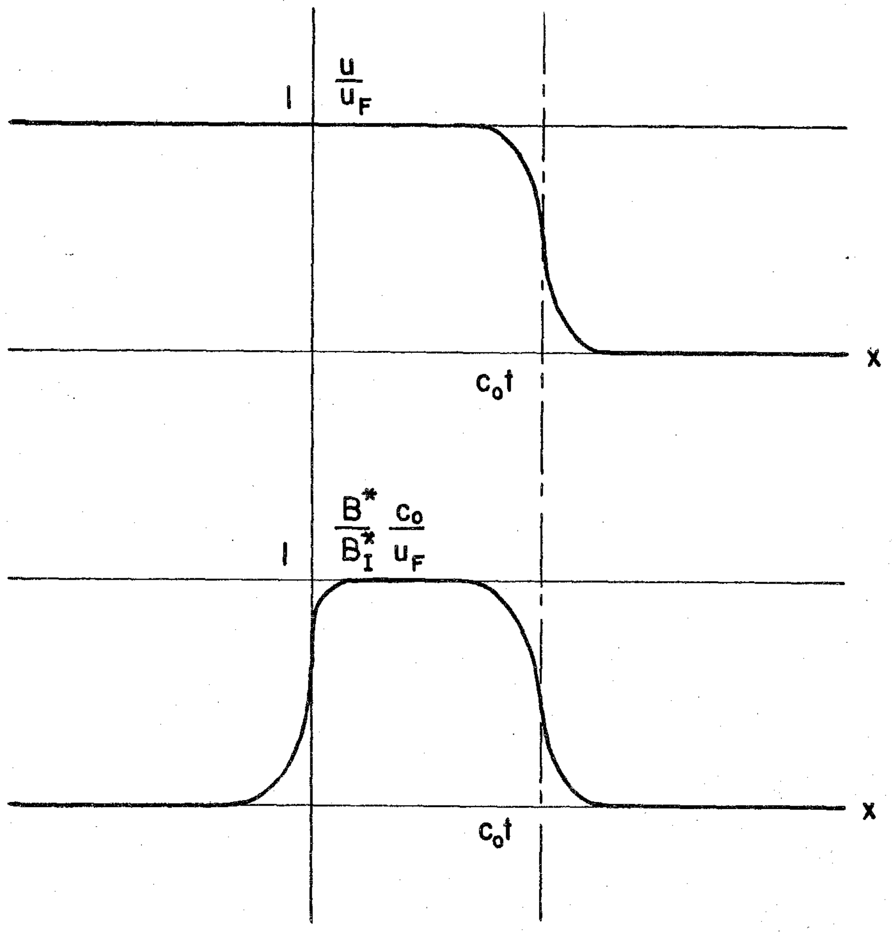
1
$\frac{u}{u_F}$
$c_0 t$
x
1
$\frac{B^*}{B_I^*} \frac{c_0}{u_F}$
$c_0 t$
x

FIG. 3

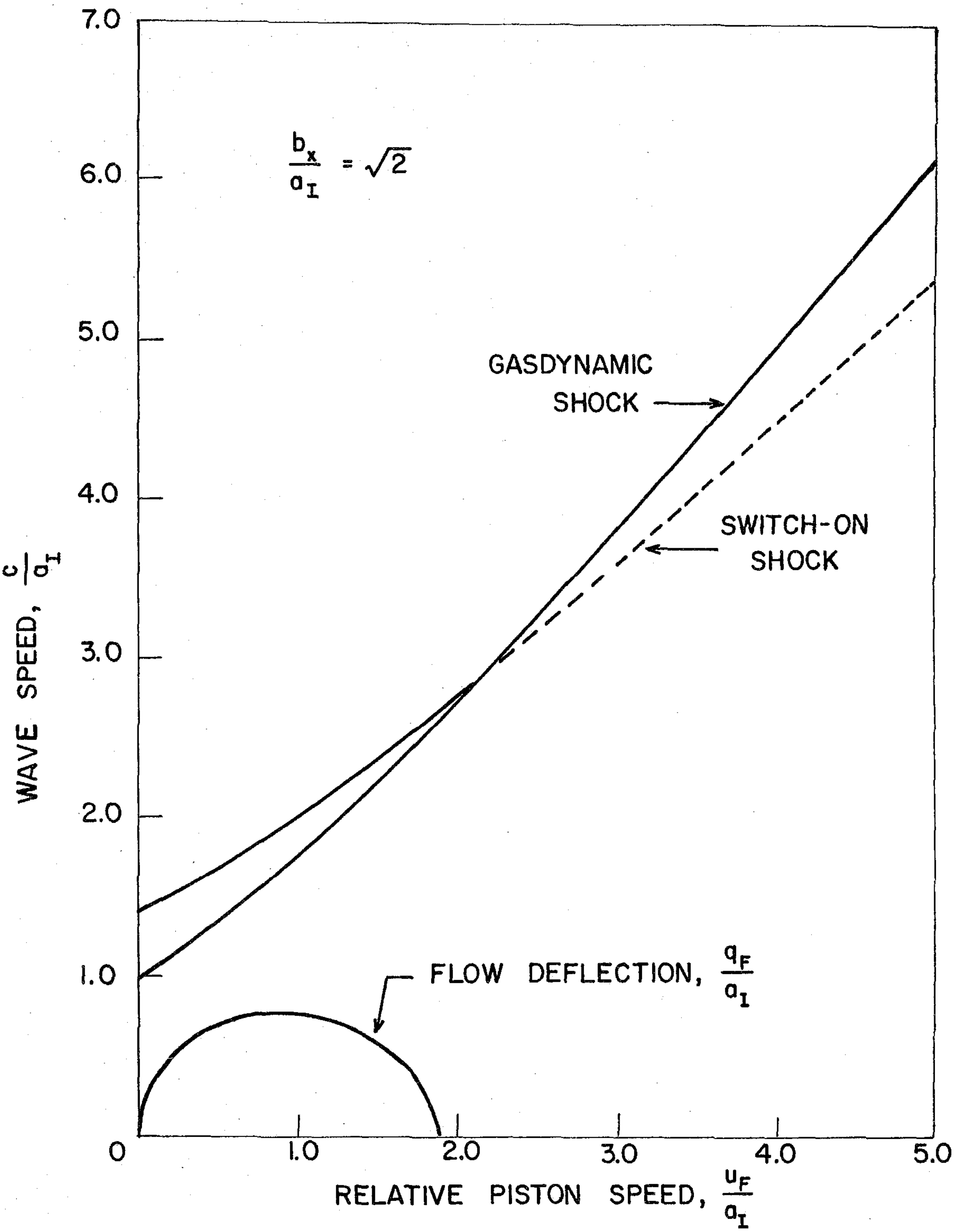
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
$\frac{b_x}{a_I} = \sqrt{2}$
GASDYNAMIC SHOCK
SWITCH-ON SHOCK
FLOW DEFLECTION, $\frac{q_F}{a_I}$
WAVE SPEED, $\frac{c}{a_I}$
1.0
2.0
3.0
4.0
5.0
RELATIVE PISTON SPEED, $\frac{u_F}{a_I}$

FIG. 4

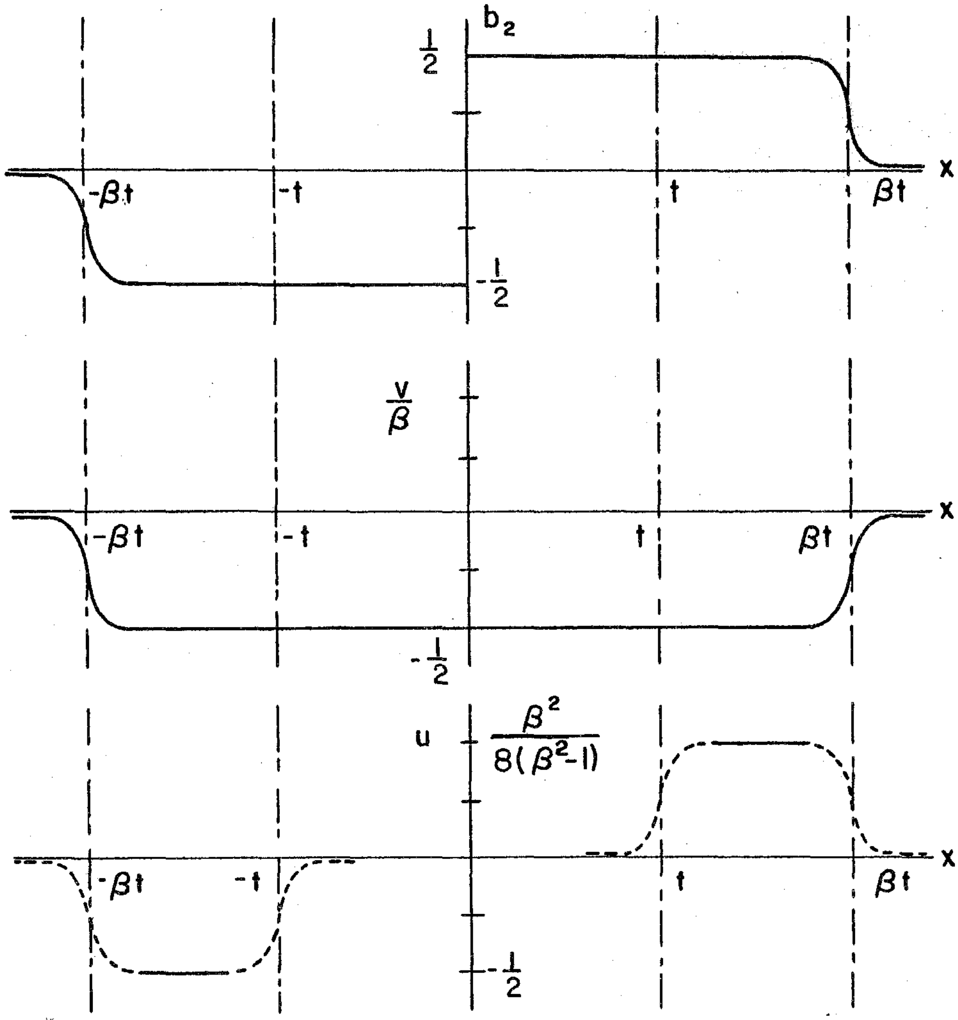

FIG. 5

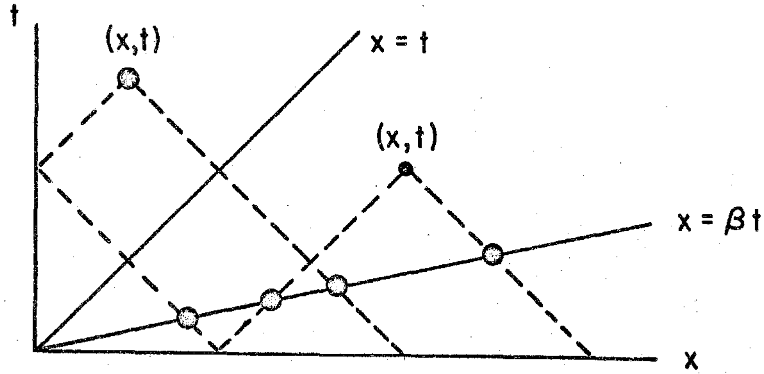

FIG. 6

UNCLASSIFIED

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12. Abstract:

Shock waves in an infinitely conducting fluid are studied by means of an idealized piston problem. Switch-on waves are shown to be associated with the discharge of a current sheet. The effect of finite conductivity is studied for both ordinary and switch-on waves. It is shown how current sheets are diffused about the wave front. The effects of non-linearity are discussed in a qualitative way.